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Free resolutions in multivariable operator theory ☆

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Abstract

Let $\mathscr{A}_d$ be the complex polynomial ring in d variables. A contractive $\mathscr{A}_d$-module is Hilbert space $\mathscr{H}$ equipped with an $\mathscr{A}_d$ action such that for any $\xi_1, \xi_2, \ldots, \xi_d \in \mathscr{H}$,

$$||z_1\xi_1 + z_2\xi + \cdots + z_d\xi_d||^2 \leqslant ||\xi_1||^2 + ||\xi_2||^2 + \cdots + ||\xi_d||^2.$$

Such objects have been shown to be useful for modeling d-tuples of mutually commuting operators acting on a Hilbert space. There is a subclass of the category of contractive $\mathscr{A}_d$ modules whose members play the role of free objects. Given a contractive $\mathscr{A}_d$-module, one can construct a free resolution, i.e. an exact sequence of partial isometries of the following form:

$$\cdots \overset{\Phi_2}{\to} \mathscr{F}_1 \overset{\Phi_1}{\to} \mathscr{F}_0 \overset{\Phi_0}{\to} \mathscr{H} \to 0, \qquad (*)$$

where $\mathscr{F}_i$ is a free module for each $i \geqslant 0$. The notion of a localization of a free resolution will be defined, in which for each $\lambda \in B_d$ there is a vector space complex of linear maps derived from $(*)$:

$$\cdots \overset{\Phi_3(\lambda)}{\to} \mathscr{C}_2 \overset{\Phi_2(\lambda)}{\to} \mathscr{C}_1 \overset{\Phi_1(\lambda)}{\to} \to \mathscr{C}_0.$$

We shall show that the homology of this complex is isomorphic to the homology of the Koszul complex of the d-tuple $(\varphi^1, \varphi^2, \ldots, \varphi^d)$, of where φ^i is the ith coordinate function of a Möbius transform on B_d such that $\varphi(\lambda) = 0$.

☆Theorem 2.11 originally appeared in authors Ph.D. thesis which was completed at U.C. Berkeley under the supervision of William Arveson. The shortened proof which appears in this paper was inspired by a useful discussion with David Eisenbud regarding homological issues.

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1. Introduction

Let $\mathscr{A}_d = \mathbb{C}[z_1, z_2, \dots, z_d]$ be the polynomial ring in d variables. We define a *contractive Hilbert $\mathscr{A}_d$-module* to be a module $\mathscr{H}$ over $\mathscr{A}_d$ that is a Hilbert space and that has the additional property that for all $\xi_1, \xi_2, \dots, \xi_d \in \mathscr{H}$,

$$\left\| \sum_{k=1}^{d} z_k \xi_k \right\|^2 \leqslant \sum_{k=1}^{d} ||\xi_k||^2.$$

Obviously, any closed submodule $\mathscr{K}$ of a contractive $\mathscr{A}$-module $\mathscr{H}$ is a contractive $\mathscr{A}$-module. Consider the Banach space quotient $\mathscr{H}/\mathscr{K}$. One can identify this space with $\mathscr{H} \ominus \mathscr{K}$, and define a contractive $\mathscr{A}_d$-module structure on it by compressing polynomials by $P_{\mathscr{H} \ominus \mathscr{K}}$.

The notion of a Hilbert module was used by Arveson [3] to represent commuting d-contractions of operators. Indeed, the actions of $z_1, z_2, \dots, z_d$ on $\mathscr{H}$ correspond to a mutually commuting d-tuple of linear operators $(T_1, T_2, \dots, T_d)$ by defining $z_i \xi = T_i \xi$ for all $\xi \in \mathscr{H}$. The contractive condition on the module $\mathscr{H}$ is equivalent to saying that the row operator $(T_1 T_2 \cdots T_d)$ is contractive when seen as a map from $\mathscr{H} \oplus \mathscr{H} \oplus \cdots \oplus \mathscr{H} \to \mathscr{H}$. Conversely, given a d-tuple of linear operators $(T_1, T_2, \dots, T_d)$ acting on a Hilbert space $\mathscr{H}$ where the components mutually commute and the row operator $(T_1 T_2 \cdots T_d)$ is contractive, one can define a contractive $\mathscr{A}_d$-module structure on $\mathscr{H}$ by defining $z_i \xi = T_i \xi$ for each $\xi \in \mathscr{H}$. Given a contractive $\mathscr{A}_d$-module $\mathscr{H}$, we call the d-tuple $(T_1, T_2, \dots, T_d)$ the *associated d-tuple of $\mathscr{H}$*.

An example of a contractive $\mathscr{A}_d$-module, and one which plays an important role in the theory, is the following: Let H_d^2 be the Hilbert space of holomorphic functions on the unit ball B_d of $\mathbb{C}^d$ derived from the following reproducing kernel:

$$k(z, \lambda) = \frac{1}{1 - \langle z, \lambda \rangle_{\mathbb{C}^d}}, \quad z, \lambda \in B_d. \tag{1.1}$$

The function $k_\lambda : z \mapsto k(z, \lambda)$ is in H_d^2 and in fact for any $\xi \in H_d^2$,

$$\xi(\lambda) = \langle \xi, k_\lambda \rangle.$$

A contractive $\mathscr{A}_d$-module structure is defined on H_d^2 as follows: For any $\xi \in H_d^2$, $p(z_1, z_2, \dots, z_d) \in \mathscr{A}_d$, $p(z_1, z_2, \dots, z_d)\xi$ is simply the function $p(z_1, z_2, \dots, z_d)$ multiplied by ξ. The Hilbert space direct sum of n copies of H_d^2 is also a contractive $\mathscr{A}_d$-module and can be expressed by $H_d^2 \otimes \mathscr{C}$, where $\mathscr{C}$ is a Hilbert space of dimension n.

Another simple example of a contractive $\mathscr{A}_d$-module arises as follows: Let $C^*(\partial B_d)$ be the C^*-algebra of continuous functions on the unit sphere in $\mathbb{C}^d$. Let $\pi : C^*(\partial B_d) \to \mathscr{B}(\mathscr{K})$ be a *-representation. Then polynomials act on $\mathscr{K}$ in the natural way, and $\mathscr{H}$ becomes a contractive $\mathscr{A}_d$-module. Such modules are called *spherical modules*.

Direct sums of the form $(H^2_d \otimes \mathscr{D}) \oplus \mathscr{S}$, where $\mathscr{D}$ is a Hilbert space and $\mathscr{S}$ is a spherical module, play the role of universal objects in the category of $\mathscr{A}_d$-modules. The following result is due to Arveson [2].

Theorem 1.1. *Let $\mathscr{H}$ be a contractive $\mathscr{A}_d$-module. There exists a Hilbert space $\mathscr{D}$, a spherical module $\mathscr{S}$, and a module homomorphism $U : (H^2_d \otimes \mathscr{D}) \oplus \mathscr{S} \to \mathscr{H}$ such that $UU^* = 1$, i.e. U is a coisometry.*

A uniqueness condition also applies. First we state the following definition:

Definition 1.2. Let $\mathscr{H}$ be a contractive $\mathscr{A}_d$-module, let $\mathscr{D}$ be a Hilbert space, let $\mathscr{S}$ be a spherical module, and let $U : (H^2_d \otimes \mathscr{D}) \oplus \mathscr{S} \to \mathscr{H}$ be a coisometry. The triple $(\mathscr{D}, \mathscr{S}, U)$ is said to be a minimal dilation of $\mathscr{H}$ if the closed submodule of $(H^2_d \otimes \mathscr{D}) \oplus \mathscr{S}$ generated by $U^*\mathscr{H}$ is $(H^2_d \otimes \mathscr{D}) \oplus \mathscr{S}$.

The following theorem, which is also due to Arveson [2], states that any two minimal dilations are naturally isomorphic.

Theorem 1.3. *Every contractive $\mathscr{A}_d$-module $\mathscr{H}$ has a minimal dilation. Furthermore, if $(\mathscr{D}, \mathscr{S}, U)$ and $(\mathscr{D}', \mathscr{S}', U')$ are minimal dilations, then there is a unitary module isomorphism $V : (H^2_d \otimes \mathscr{D}) \oplus \mathscr{S} \to (H^2_d \otimes \mathscr{D}') \oplus \mathscr{S}'$ such that $U = U'V$. Furthermore, V has the form $V = (1_{H^2_d} \otimes W) \oplus W'$.*

Let $\mathscr{C}$ be a Hilbert space. The free module $H^2_d \otimes \mathscr{C}$ can be viewed as a space of (weakly) holomorphic $\mathscr{C}$-valued functions defined on B_d. Indeed, if $\xi \in H^2_d \otimes \mathscr{C}$ and $\lambda \in B_d$, then $\xi(\lambda)$ is defined to be the unique element of $\mathscr{C}$ such that for all $\eta \in \mathscr{C}$,

$$\langle \xi, k_\lambda \otimes \eta \rangle = \langle \xi(\lambda), \eta \rangle_{\mathscr{C}}.$$

This identification of $H^2_d \otimes \mathscr{C}$ with a space of vector-valued holomorphic functions gives us a useful way of perceiving module homomorphisms between tensor multiples of H^2_d. Indeed, let $\mathscr{D}$ and $\mathscr{C}$ be Hilbert spaces, and let $\Phi : H^2_d \otimes \mathscr{D} \to H^2_d \otimes \mathscr{C}$ be a module homomorphism. For each $\lambda \in B_d$, define the bounded linear operator $\Phi(\lambda) : \mathscr{D} \to \mathscr{C}$ by $\Phi(\lambda)\eta = \Phi(1 \otimes \eta)(\lambda)$. Since Φ is a homomorphism, it follows that for all $\xi \in H^2_d \otimes \mathscr{D}$ and $\lambda \in B_d$, $(\Phi\xi)(\lambda) = \Phi(\lambda)\xi(\lambda)$. Hence, Φ is given by pointwise multiplication by the operator-valued function $\Phi(\lambda)$. From the definition, it is not difficult to show that the adjoint of Φ has the following property: Let $\lambda \in B_d$, and let $\eta \in \mathscr{C}$. Then

$$\Phi^*(k_\lambda \otimes \eta) = k_\lambda \otimes \Phi(\lambda)^* \eta. \tag{1.2}$$

As a consequence of (1.2), we have the following boundedness condition on $\Phi(\lambda)$:

$$\|\Phi(\lambda)\| \leqslant \|\Phi\| \quad \forall \lambda \in B_d.$$

Thus, the function $\lambda \in B_d \mapsto \Phi(\lambda)$ is a bounded $\mathscr{B}(\mathscr{D}, \mathscr{C})$-valued (weakly) holomorphic function on B_d. For an arbitrary free module homomorphism $\Phi : (H_d^2 \otimes \mathscr{D}) \oplus \mathscr{S} \to (H_d^2 \otimes \mathscr{C}) \oplus \mathscr{S}'$, one defines the operator $\Phi(\lambda)$ to be the localization to λ of the homomorphism $\Phi' = P_{H_d^2 \otimes \mathscr{C}} \Phi \restriction_{H_d^2 \otimes \mathscr{D}} : H_d^2 \otimes \mathscr{D} \to H_d^2 \otimes \mathscr{C}$.

We now detail the construction of a free resolution of a contractive $\mathscr{A}_d$ module $\mathscr{H}$. By Theorem 1.1, there is a free module $\mathscr{F}_0$ and a coisometric module homomorphism $\Phi_0 : \mathscr{F}_0 \to \mathscr{H}$. The kernel of Φ_0 is a closed submodule of $\mathscr{F}_0$, and one can apply Theorem 1.1 again to obtain a free module $\mathscr{F}_1$ and a partially isometric module homomorphism $\Phi_1 : \mathscr{F}_1 \to \mathscr{F}_0$ with image $\ker \Phi_0$. By repeating this process, one obtains an exact sequence as follows:

$$\cdots \xrightarrow{\Phi_2} \mathscr{F}_1 \xrightarrow{\Phi_1} \mathscr{F}_0 \xrightarrow{\Phi_0} \mathscr{H} \to 0, \tag{1.3}$$

where $\mathscr{F}_i$ is free for each i. We call this a free resolution of $\mathscr{H}$, and it is analogous to the free resolution in the theory of finitely generated modules over Noetherian rings.

Each $\mathscr{F}_i$ appearing in (1.3) has the form $(H_d^2 \otimes \mathscr{C}_i) \oplus \mathscr{S}_i$ for some Hilbert space $\mathscr{C}_i$ and spherical module $\mathscr{S}_i$. If we localize to a point $\lambda \in B_d$, we obtain the following sequence of linear maps:

$$\cdots \xrightarrow{\Phi_3(\lambda)} \mathscr{C}_2 \xrightarrow{\Phi_2(\lambda)} \mathscr{C}_1 \xrightarrow{\Phi_1(\lambda)} \mathscr{C}_0. \tag{1.4}$$

Since (1.3) is exact, it follows that (1.4) is a complex for each λ.

The main result of Section 2 is that the homology of (1.4) at 0 is closely connected to the spectral properties of the d-tuple $(T_1, T_2, \dots, T_d)$ associated with $\mathscr{H}$. By "spectral properties" we are referring to the spectrum of a d-tuple of operators in the sense of Taylor (cf. [10]). The following discussion summarizes this more precisely: For each $k \in \mathbb{Z}$, let $\bigwedge^k \mathbb{C}^d$ be the k-fold alternating product of $\mathbb{C}^d$, taken to be the trivial vector space when $k < 0$. For each k, let $E_k = \mathscr{H} \otimes \bigwedge^k \mathbb{C}^d$. Fix an orthonormal basis $\{e_1, e_2, \dots, e_d\}$ for $\mathbb{C}^d$, and for each k, define the linear map $\partial_k : E_k \to E_{k-1}$ as follows:

$$\partial_k(\xi \otimes e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{j=1}^{k} (-1)^{j+1} z_{i_j} \xi \otimes e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k}.$$

A straightforward calculation shows that the following sequence is a complex:

$$0 \to E_d \xrightarrow{\partial_d} E_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} E_0 \to 0. \tag{1.5}$$

What we will show in Section 2 is that

$$\frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}} \cong \frac{\ker \Phi_k(0)}{\operatorname{im} \Phi_{k+1}(0)}.$$

Hence the homology of (1.5) and that of (1.4) when $\lambda = 0$ are identical.

In Section 3, we introduce the notion of a Möbius transform of a contractive $\mathscr{A}_d$-module. We recall that a Möbius transform φ on B_d is a bijective holomorphism from B_d onto itself. If $\mathscr{H}$ is a pure contractive $\mathscr{A}_d$-module, and $(T_1, T_2, \ldots, T_d)$ is the associated d-tuple, then we can define the Möbius transform $(\mathscr{H})_\varphi$ of $\mathscr{H}$ to be the contractive $\mathscr{A}_d$-module where the underlying Hilbert space of $(\mathscr{H})_\varphi$ is the underlying Hilbert space of $\mathscr{H}$, but $z_1, z_2, \ldots, z_d$ act as $\varphi^1, \varphi^2, \ldots, \varphi^d$, respectively, where φ^i is the ith coordinate of φ. In the case where $\mathscr{H} = H^2_d$, we will show that there exists a unitary module isomorphism $U_\varphi : H^2_d \to (H^2_d)_\varphi$, which carries the space of all functions in H^2_d that vanish at 0 to the space of all functions that vanish at λ where $\phi(\lambda) = 0$. We will also show that the set of these U_φ act ergodically on H^2_d in the sense that no proper nontrivial closed submodule of H^2_d is invariant under U_φ for all Möbius transforms φ.

Finally, in Section 4, we use the results on Section 3 to describe the homology of (1.4) for an arbitrary $\lambda \in B_d$. In particular, Theorem 4.3 states that the homology of the Koszul complex of $(\mathscr{H})_\varphi$ is equivalent to the localized complex of (1.4). Hence (1.4) contains spectral information (in the sense of Taylor) of the Möbius transformation on $\mathscr{H}$.

2. The Koszul complex and free resolution of a contractive $\mathscr{A}_d$-module

In his seminal paper [10], Taylor defined the notion of invertibility of d-tuples of commuting operators acting on a Banach space $\mathscr{B}$. We will briefly summarize Taylor's construction:

Let $\mathscr{B}$ be a Banach space, and let $(a_1, a_2, \ldots, a_d)$ be a d-tuple of mutually commuting bounded operators on $\mathscr{B}$. For $k \in \mathbb{Z}$, let $\bigwedge^k \mathbb{C}^d$ be the k-fold wedge product of $\mathbb{C}^d$, where this is taken to be the trivial vector space when $k < 0$. Let $E_k = \mathscr{B} \otimes \bigwedge^k \mathbb{C}^d$. Note that E_k can be viewed as an $\binom{d}{k}$-fold direct sum of $\mathscr{B}$, hence it is itself a Banach space. Fix an orthonormal basis $\{e_1, e_2, \ldots, e_d\}$ for $\mathbb{C}^d$. For each k, define $\partial_k : E_k \to E_{k-1}$ by

$$\partial_k(\xi \otimes e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{j=1}^{d} (-1)^{j+1} a_j \xi \otimes e_{i_1} \wedge \cdots e_{i_j} \cdots e_{i_k}.$$

A simple computation shows that $\partial_{k-1}\partial_k = 0$. Hence, we have the following complex of Banach spaces:

$$\cdots \to 0 \to E_d \xrightarrow{\partial_d} E_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} E_0 \xrightarrow{\partial_0} 0. \tag{2.1}$$

The following is Taylor's definition of invertibility:

Definition 2.1. A d-tuple of mutually commuting operators acting on a common Banach space $\mathscr{B}$ is said to be invertible if the sequence in (2.1) is exact.

One can immediately provide a definition of the spectrum of $(a_1, a_2, \dots, a_d)$:

Definition 2.2. Let $a = (a_1, a_2, \dots, a_d)$ be a d-tuple of mutually commuting operators on a Banach space $\mathscr{B}$. Then spec (a) is defined to be the set $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d) \in \mathbb{C}^d$ where $(a_1 - \lambda_1, a_2 - \lambda_2, \dots, a_d - \lambda_d)$ is not invertible.

We remark that in the case where $d = 1$, both Definitions 2.1 and 2.2 reduce to the usual definitions of invertibility and spectrum for single operators.

If we restrict ourselves to the case where $\mathscr{B}$ is a Hilbert space $\mathscr{H}$, and the d-tuple $(T_1, T_2, \dots, T_d)$ consists of mutually commuting bounded operators on $\mathscr{H}$, then we can express homological features of (2.1) in terms of a single self-adjoint operator. Our presentation will follow Arveson [1], but this idea appeared earlier in the work of such authors as Curto [5] and Vasilescu [11].

Let $\mathscr{H}$ and $(T_1, T_2, \dots, T_d)$ be as above, and let $\{e_1, e_2, \dots, e_d\}$ be our fixed orthonormal basis for $\mathbb{C}^d$. The space $\bigwedge^k \mathbb{C}^d$ has a natural inner product defined by

$$\langle z_1 \wedge z_2 \wedge \cdots \wedge z_k, w_1 \wedge w_2 \wedge \cdots \wedge w_k \rangle = \det(\langle z_i, w_j \rangle)_{ij}, z_i, w_i \in \mathbb{C}^d.$$

Hence the spaces E_k are tensor products of Hilbert spaces. Let $\bigwedge \mathbb{C}^d = \bigoplus_{k \in \mathbb{Z}} \bigwedge^k \mathbb{C}^d$. Define E to be the direct sum of the E_k's, i.e.

$$E = \bigoplus_{k \in \mathbb{Z}} E_k = \mathscr{H} \otimes \bigwedge \mathbb{C}^d.$$

Let $c_1, c_2, \dots, c_d$ be the operators on $\bigwedge \mathbb{C}^d$ defined by

$$c_i(z_1 \wedge z_2 \wedge \cdots \wedge z_k) = e_i \wedge z_1 \wedge \cdots \wedge z_k, \quad z_i \in \mathbb{C}^d.$$

We then define the linear operator $\partial : E \to E$ to be the sum

$$T_1 \otimes c_1^* + T_2 \otimes c_2^* + \cdots + T_d \otimes c_d^*.$$

One then checks that the restriction of ∂ to E_k is ∂_k. One can now prove the following theorem (cf. [1]):

Theorem 2.3. *Let $(T_1, T_2, \dots, T_d)$ be a d-tuple of mutually commuting operators on a common Hilbert space $\mathscr{H}$. If ∂ is the corresponding boundary operator, then $(T_1, T_2, \dots, T_d)$ is invertible iff $\partial + \partial^*$ is invertible.*

The operator $\partial + \partial^*$ is called the Dirac operator corresponding to $(T_1, T_2, \dots, T_d)$. This idea of expressing the invertibility of a d-tuple in terms of single operator suggests the following definition:

Definition 2.4. A d-tuple of operators as in Theorem 2.3 is said to be Fredholm if the corresponding Dirac operator $\partial + \partial^*$ is Fredholm.

We note that in the case where $d = 1$, this definition corresponds to the usual definition of Fredholmness via an easy application of Atkinsons' equivalences (see [4], for example).

The Fredholmness of a Dirac operator has an important consequence with respect to the homology of the Koszul complex. This is expressed in the following theorem, which follows from the definition of ∂ by a straightforward argument.

Theorem 2.5. *A d-tuple $(T_1, T_2, \dots, T_d)$ is Fredholm iff the homology spaces $\ker \partial_k / \mathrm{im}\, \partial_{k+1}$ are finite dimensional for all $k \in \mathbb{Z}$.*

Theorem 2.5 allows us to generalize the notion of index to Fredholm d-tuples. Indeed, the index of a Fredholm d-tuple $(T_1, T_2, \dots, T_d)$ is defined to be the alternating sum of the dimensions of the homology spaces of the Koszul complex, i.e.

$$\mathrm{index}(T_1, T_2, \dots, T_d) = \sum_{k \in \mathbb{Z}} (-1)^{k+1} \dim \frac{\ker \partial_k}{\mathrm{im}\, \partial_{k+1}}.$$

Again, we note that this definition reduces to the usual definition of index when one takes $d = 1$.

We now show that the d-tuple associated with the free Hilbert module H_d^2 is Fredholm and we compute its homology.

Theorem 2.6. *The d-tuple $(S_1, S_2, \dots, S_d)$ associated with H_d^2 is Fredholm. Furthermore, the extended sequence of maps*

$$\cdots \overset{\partial_2}{\to} E_1 \overset{\partial_1}{\to} E_0 \overset{\partial_{1/2}}{\to} \mathbb{C} \to 0,$$

where $\partial_{1/2}$ is defined to be the evaluation map $\xi \mapsto \xi(0)$, is exact.

A proof of the fact the $(S_1, S_2, \dots, S_d)$ is Fredholm can be found in [1]. We will rely on this fact to prove the remainder of the theorem.

Proof. Let k be an integer no less than 1. By Theorem 2.5, the space $\mathrm{im}\, \partial_{k+1}$ has finite codimension in $\ker \partial_k$. It is a standard fact in operator theory that if the image of a bounded operator has finite codimension in a larger closed subspace, then the image is closed. Hence, $\mathrm{im}\, \partial_{k+1}$ is a closed subspace of finite codimension in $\ker \partial_k$. We now show that $\mathrm{im}\, \partial_{k+1} = \ker \partial_k$. To this end, let $\xi \in \ker \partial_k$. Then ξ can be written as follows:

$$\sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant d} \xi_{i_1,i_2,\ldots,i_k} \otimes e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$$

where $\xi_{i_1,i_2,\ldots,i_k} \in H_d^2$. Supposing for the moment that these $\xi_{i_1,i_2,\ldots,i_k}$ are homogeneous polynomials all of the same degree N, then $\partial_k \xi$ is in a similar form but with common degree $N+1$. It follows in the general case that if $\xi^n_{i_1,i_2,\ldots,i_k}$ is the nth degree homogeneous component of $\xi_{i_1,i_2,\ldots,i_k}$, then

$$\xi^n := \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant d} \xi^n_{i_1,i_2,\ldots,i_k} \otimes e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \in \ker \partial_k.$$

By Corollary 17.5 in [7], $\xi^n \in \operatorname{im} \partial_{k+1}$ for each n, hence $\xi = \sum_{n=0}^{\infty} \xi_n \in \operatorname{im} \partial_{k+1}$.

We proceed to the case where $k = 0$. By Theorem 3.1, the following row operator is a partial isometry with image $\{\xi \in H_d^2 : \xi(0) = 0\}$:

$$(S_1 \;\; S_2 \cdots S_d) : \overbrace{H_d^2 \oplus \cdots \oplus H_d^2}^{d \text{ times}} \to H_d^2. \tag{2.2}$$

A generic element ζ of E_1 has the following form:

$$\sum_{k=1}^{d} \xi_k \otimes e_k, \quad \xi_k \in H_d^2.$$

Hence $\partial_1 \xi = \sum_{k=1}^{d} S_k \xi_k$. It follows from (2.2) and the statement preceding it that $\operatorname{im} \partial_1 = \ker \partial_{1/2}$.

The subjectivity of $\partial_{1/2}$ is clear. $\square$

Corollary 2.7. *Let*

$$\cdots \xrightarrow{\partial_2} E_1 \xrightarrow{\partial_1} E_0 \to 0$$

be the Koszul complex of $(S_1, S_2, \ldots, S_d)$. *Then its homology is as follows*:

$$\frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}} = 0, \quad k \geqslant 1,$$

$$\frac{\ker \partial_0}{\operatorname{im} \partial_1} \cong \mathbb{C}.$$

Corollary 2.8. *Let* $(S_1', S_2', \ldots, S_d')$ *be the d-tuple associated with a free module* $\mathscr{F} = H_d^2 \otimes \mathscr{C}$ *with the following extended sequence of maps*:

$$\cdots \xrightarrow{\partial_2} E_1 \xrightarrow{\partial_1} E_0 \xrightarrow{\partial_{1/2}} \mathscr{C} \to 0.$$

Then for $k \geq 1$, $\ker \partial_k = \operatorname{im} \partial_{k+1}$ *and* $\ker \partial_0 / \operatorname{im} \partial_1 \cong \ker \partial_0 \cap (\operatorname{im} \partial_1)^\perp \cong \mathscr{C}$. *In other words* $\operatorname{im} \partial_{k+1}$ *is closed in* $\ker \partial_k$, *and in the case where* $k = 1$, *the codimension of* $\operatorname{im} \partial_{k+1}$ *in* $\ker \partial_k$ *is* $\dim \mathscr{C}$.

Theorem 2.9. *Let* $(Z_1, Z_2, \ldots, Z_d)$ *be the d-tuple associated with a spherical module* $\mathscr{S}$. *Then the Koszul complex of* $(Z_1, Z_2, \ldots, Z_d)$ *is exact.*

Proof. This follows from [10, Lemma 1.1], using the fact that the Z_i's are normal and $\sum_{i=1}^d Z_i Z_i^* = 1$. $\square$

Corollary 2.10. *Let* $\mathscr{F} = (H_d^2 \otimes \mathscr{C}) \oplus \mathscr{S}$ *be a free module. Let*

$$\cdots \xrightarrow{\partial_3} E_2 \xrightarrow{\partial_2} E_1 \xrightarrow{\partial_1} E_0 \to 0$$

be the Koszul complex of $\mathscr{F}$. *Define* $\partial_{1/2} : E_0 \to \mathscr{C}$ *to be the evaluation map* $\xi \mapsto (P_{H_d^2 \otimes \mathscr{C}} \xi)(0)$. *Then the following sequence is exact*:

$$\cdots \xrightarrow{\partial_2} E_1 \xrightarrow{\partial_1} E_0 \xrightarrow{\partial_{1/2}} \mathscr{C} \to 0.$$

Naturally, our entire discussion on Koszul complexes of d-tuples of operators can be rephrased in terms of $\mathscr{A}_d$-modules. Indeed, one simply takes $\mathscr{H}$ to be the module defined by $z_i \xi = T_i \xi$ for all $\xi \in \mathscr{H}$. The spaces E_k are defined analogously, with $\partial_k : E_k \to E_{k-1}$ reexpressed as

$$\partial_k(\xi \otimes e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) = \sum_{j=1}^{k} (-1)^{j+1} z_{i_j} \xi \otimes e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k}.$$

Since $\mathscr{A}_d$ is a commutative algebra, the maps ∂_k are module homomorphisms. Hence, sequence (2.1) can be viewed as a complex of $\mathscr{A}_d$-modules.

As summarized in Section 1, if one starts with a contractive $\mathscr{A}_d$-module $\mathscr{H}$, then by means of dilation theory one may construct a free resolution of $\mathscr{H}$:

$$\cdots \xrightarrow{\Phi_2} \mathscr{F}_1 \xrightarrow{\Phi_1} \mathscr{F}_0 \xrightarrow{\Phi_0} \mathscr{H} \to 0, \tag{2.3}$$

which is an exact sequence where the Φ_i's are partial isometries and the $\mathscr{F}_i$'s are free modules. One then localizes (2.3) to a point $\lambda \in B_d$ to obtain a complex of vector spaces:

$$\cdots \xrightarrow{\Phi_3(\lambda)} \mathscr{C}_2 \xrightarrow{\Phi_2(\lambda)} \mathscr{C}_1 \xrightarrow{\Phi_1(\lambda)} \mathscr{C}_0.$$

The main result of this section is the following:

Theorem 2.11. *Let $\mathscr{H}$ be a contractive $\mathscr{A}_d$-module, and let*

$$\cdots \overset{\Phi_2}{\to} \mathscr{F}_1 \overset{\Phi_1}{\to} \mathscr{F}_0 \overset{\Phi_0}{\to} \mathscr{H} \to 0 \tag{2.4}$$

be a free resolution of $\mathscr{H}$. Localize at 0 to obtain the following complex:

$$\cdots \overset{\Phi_3(0)}{\to} \mathscr{C}_2 \overset{\Phi_2(0)}{\to} \mathscr{C}_1 \overset{\Phi_1(0)}{\to} \mathscr{C}_0.$$

Let

$$\cdots \overset{\partial_3}{\to} E_2 \overset{\partial_2}{\to} E_1 \overset{\partial_1}{\to} E_0 \to 0$$

be the Koszul complex of $\mathscr{H}$. Then for all $k \geqslant 1$,

$$\frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}} \cong \frac{\ker \Phi_k(0)}{\operatorname{im} \Phi_{k+1}(0)}.$$

Proof. In the course of this proof, we will use $\mathscr{F}_i^k$ and $\mathscr{H}^k$ to denote, respectively, $\mathscr{F}_i \otimes \bigwedge^k \mathbb{C}^d$ and $\mathscr{H} \otimes \bigwedge^k \mathbb{C}^d$. Since for any k, $\bigwedge^k \mathbb{C}^d$ is finite dimensional, tensoring the components of (2.4) by $\bigwedge^k \mathbb{C}^d$ preserves exactness. Hence the following sequence is exact:

$$\cdots \overset{\Phi_2 \otimes 1_{\wedge^k \mathbb{C}^d}}{\to} \mathscr{F}_1^k \overset{\Phi_1 \otimes 1_{\wedge^k \mathbb{C}^d}}{\to} \mathscr{F}_0^k \overset{\Phi_0 \otimes 1_{\wedge^k \mathbb{C}^d}}{\to} \mathscr{H}^k \to 0.$$

For the sake of convenience, and since it will cause no confusion in what follows, we will denote maps of the form $A \otimes \bigwedge^k \mathbb{C}^d$ by A for any linear operator A on a Hilbert space $\mathscr{K}$. Furthermore, unless otherwise stated, we shall use ∂_k to denote *any* Koszul complex mapping $\mathscr{M} \otimes \bigwedge^k \mathbb{C}^d \to \mathscr{M} \otimes \bigwedge^{k-1} \mathbb{C}^d$.

We claim that the following diagram commutes, and, with the exception of the $\mathscr{C}_*$ column, is exact on rows and columns.

$$\begin{tikzcd}
 & \vdots \arrow[d, "\Phi_{i+3}"'] & \vdots \arrow[d, "\Phi_{i+3}"'] & \vdots \arrow[d, "\Phi_{i+3}(0)"'] & \\
\cdots \arrow[r, "\partial_2"] & \mathcal{F}^1_{i+2} \arrow[r, "\partial_1"] \arrow[d, "\Phi_{i+2}"'] & \mathcal{F}^0_{i+2} \arrow[r, "\partial_{1/2}"] \arrow[d, "\Phi_{i+2}"'] & \mathcal{C}_{i+2} \arrow[r] \arrow[d, "\Phi_{i+2}(0)"'] & 0 \\
\cdots \arrow[r, "\partial_2"] & \mathcal{F}^1_{i+1} \arrow[r, "\partial_1"] \arrow[d, "\Phi_{i+1}"'] & \mathcal{F}^0_{i+1} \arrow[r, "\partial_{1/2}"] \arrow[d, "\Phi_{i+1}"'] & \mathcal{C}_{i+1} \arrow[r] \arrow[d, "\Phi_{i+1}(0)"'] & 0 \\
\cdots \arrow[r, "\partial_2"] & \mathcal{F}^1_{i} \arrow[r, "\partial_1"] \arrow[d, "\Phi_{i}"'] & \mathcal{F}^0_{i} \arrow[r, "\partial_{1/2}"] \arrow[d, "\Phi_{i}"'] & \mathcal{C}_{i} \arrow[r] \arrow[d, "\Phi_{i}(0)"'] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{2.5}$$

Firstly, we check commutativity for the square

$$\begin{array}{ccc} \mathcal{F}_{i+1}^{k} & \xrightarrow{\partial_k} & \mathcal{F}_{i+1}^{k-1} \\ {\scriptstyle \Phi_{i+1}}\big\downarrow & & {\scriptstyle \Phi_{i+1}}\big\downarrow \\ \mathcal{F}_{i}^{k} & \xrightarrow{\partial_k} & \mathcal{F}_{i}^{k-1}, \end{array} \tag{2.6}$$

where $i, k \geqslant 1$. This amounts to showing that $\partial_k \Phi_{i+1} = \Phi_{i+1} \partial_k$. Consider an element in $\mathscr{F}_{i+1}^{k}$ of the form

$$\xi \otimes e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}, \tag{2.7}$$

where $\xi \in \mathscr{F}_{i+1},\ 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant d$. Applying Φ_{i+1} and then ∂_k to this gives us

$$\sum_{j=1}^{k} (-1)^{j+1} z_{i_j} \Phi_{i+1} \xi \otimes e_{i_1} \wedge \cdots \wedge e_{i_j} \wedge \cdots e_{i_k}. \tag{2.8}$$

Since Φ_{i+1} is a module homomorphism, the z_{i_j}'s commute with Φ_{i+1}, and the resulting expression

$$\sum_{j=1}^{k} (-1)^{j+1} \Phi_{i+1} z_{ij} \xi \otimes e_{i_1} \wedge \cdots \wedge e_{i_j} \wedge \cdots \wedge e_{i_k}$$

is the result of applying ∂_k then Φ_{i+1} to (2.7). Hence (2.6) is a commuting square. For the case of squares of the following form:

$$\begin{array}{ccc} \mathcal{F}_{i+1}^{0} & \xrightarrow{\partial_{1/2}} & \mathcal{C}_{i+1} \\ {\scriptstyle \Phi_{i+1}}\big\downarrow & & {\scriptstyle \Phi_{i+1}(0)}\big\downarrow \\ \mathcal{F}_{i}^{0} & \xrightarrow{\partial_{1/2}} & \mathcal{C}_{i}, \end{array} \tag{2.9}$$

we argue as follows: First, if $\xi \in H_d^2 \otimes \mathscr{C}_{i+1} \subseteq \mathscr{F}_{i+1}$, then by definition $\partial_{1/2} \xi = \xi(0)$, hence $\Phi_{i+1}(0) \partial_{1/2} \xi = \Phi_{i+1}(0) \xi(0) = (P_{H_d^2 \otimes \mathscr{C}_{i+1}} \Phi_{i+1} \xi)(0) = \partial_{1/2} \Phi_{i+1} \xi$. For $\eta \in \mathscr{S}_{i+1}$, the spherical component of $\mathscr{F}_{i+1}$, we have $\partial_{1/2} \eta = 0$. By the exactness of (2.9), there exist elements $\eta_1, \eta_2, \ldots, \eta_d \in \mathscr{S}_{i+1}$ such that $\eta = \sum_{k=1}^{d} z_k \eta_k$. Hence $\partial_{1/2} \Phi_{i+1} \eta = \partial_{1/2} \sum_{k=1}^{d} z_k \Phi_{i+1} \eta_k$. Since the projection $P_{H_d^2 \otimes \mathscr{C}_{i+1}}$ is a module homomorphism, it follows from the definition of $\partial_{1/2}$ that $\partial_{1/2} \Phi_{i+1} \eta = 0$. Hence (2.9) commutes.

Before proceeding we make the following observation: For each $i \geqslant 0$, we denote the quotient module $\mathscr{F}_i / \Phi_{i+1}(\mathscr{F}_{i+1})$ by $\mathscr{H}_i$. Note that by assumption $\mathscr{H}$ is isomorphic to $\mathscr{H}_0$. Let

$$\cdots \overset{\partial_2'}{\to} \mathscr{H}_i^1 \overset{\partial_1'}{\to} \mathscr{H}_i^0 \to 0 \tag{2.10}$$

be the Koszul complex for $\mathscr{H}_i$. Fixing this i, we let

$$\cdots \overset{\partial_2''}{\to} \mathscr{H}_{i+1}^1 \overset{\partial_1''}{\to} \mathscr{H}_{i+1}^0 \to 0 \tag{2.11}$$

be the Koszul complex for $\mathscr{H}_{i+1}$. Note that since (2.5) commutes, the complexes (2.10) and (2.11) are induced by the maps of the Koszul complexes $\mathscr{F}_{i+1}^* \to \mathscr{F}_i^*$ and $\mathscr{F}_{i+2}^* \to \mathscr{F}_{i+1}^*$ respectively. This can be expressed by saying that the following two diagrams commute:

$$\begin{array}{ccccc}
\cdots & \xrightarrow{\partial_2} & \mathcal{F}_{i+1}^1 & \xrightarrow{\partial_1} & \mathcal{F}_{i+1}^0 \\
 & & {\scriptstyle \Phi_{i+1}}\big\downarrow & & {\scriptstyle \Phi_{i+1}}\big\downarrow \\
\cdots & \xrightarrow{\partial_2} & \mathcal{F}_i^1 & \xrightarrow{\partial_1} & \mathcal{F}_i^0 \\
 & & \big\downarrow & & \big\downarrow \\
\cdots & \xrightarrow{\partial_2'} & \mathcal{H}_i^1 & \xrightarrow{\partial_1'} & \mathcal{H}_i^0 \\
 & & \big\downarrow & & \big\downarrow \\
 & & 0 & & 0,
\end{array}$$

$$\begin{array}{ccccc}
\cdots & \xrightarrow{\partial_2} & \mathcal{F}_{i+2}^1 & \xrightarrow{\partial_1} & \mathcal{F}_{i+2}^0 \\
 & & {\scriptstyle \Phi_{i+2}}\big\downarrow & & {\scriptstyle \Phi_{i+2}}\big\downarrow \\
\cdots & \xrightarrow{\partial_2} & \mathcal{F}_{i+1}^1 & \xrightarrow{\partial_1} & \mathcal{F}_{i+1}^0 \\
 & & \big\downarrow & & \big\downarrow \\
\cdots & \xrightarrow{\partial_2''} & \mathcal{H}_{i+1}^1 & \xrightarrow{\partial_1''} & \mathcal{H}_{i+1}^0 \\
 & & \big\downarrow & & \big\downarrow \\
 & & 0 & & 0
\end{array}$$

We will use this fact to establish the following two claims:

Claim 2.12. *For $k \geqslant 2$,*

$$\frac{\ker \partial_k'}{\operatorname{im} \partial_{k+1}'} \cong \frac{\ker \partial_{k-1}''}{\operatorname{im} \partial_k''}.$$

Proof. Consider the following portion of (2.5), where $k \geqslant 2$:

$$\begin{array}{ccccccc}
\mathcal{F}_{i+2}^{k+1} & \xrightarrow{\partial_{k+1}} & \mathcal{F}_{i+2}^{k} & \xrightarrow{\partial_k} & \mathcal{F}_{i+2}^{k-1} & \xrightarrow{\partial_{k-2}} & \mathcal{F}_{i+2}^{k-2} \\
{\scriptstyle \Phi_{i+2}}\downarrow & & {\scriptstyle \Phi_{i+2}}\downarrow & & {\scriptstyle \Phi_{i+2}}\downarrow & & {\scriptstyle \Phi_{i+2}}\downarrow \\
\mathcal{F}_{i+1}^{k+1} & \xrightarrow{\partial_{k+1}} & \mathcal{F}_{i+1}^{k} & \xrightarrow{\partial_k} & \mathcal{F}_{i+1}^{k-1} & \xrightarrow{\partial_{k-1}} & \mathcal{F}_{i+1}^{k-2} \\
{\scriptstyle \Phi_{i+1}}\downarrow & & {\scriptstyle \Phi_{i+1}}\downarrow & & {\scriptstyle \Phi_{i+1}}\downarrow & & {\scriptstyle \Phi_{i+1}}\downarrow \\
\mathcal{F}_{i}^{k+1} & \xrightarrow{\partial_{k+1}} & \mathcal{F}_{i}^{k} & \xrightarrow{\partial_k} & \mathcal{F}_{i}^{k-1} & \xrightarrow{\partial_{k-1}} & \mathcal{F}_{i}^{k-2}.
\end{array} \tag{2.12}$$

Let $\zeta \in \ker \partial'_k / \mathrm{im}\, \partial'_{k+1}$. Choose a representative $\zeta_0 \in \mathscr{F}_i^k$ for ζ. By assumption, $\partial_k \zeta_0 \in \Phi_{i+1}(\mathscr{F}_{i+1}^{k-1})$. Choose $\eta_0 \in \mathscr{F}_{i+1}^k$ such that $\Phi_{i+1}\eta_0 = \partial_k \zeta_0$. By exactness of the bottom row of (2.12), $\partial_{k-1}\Phi_{i+1}\eta_0 = 0$, hence $\partial_{k-1}\eta_0 \in \ker \Phi_{i+1}$. Hence by the exactness of the last column in (2.12), η_0 defines a homology class $\eta \in \ker \partial''_{k-1}/\mathrm{im}\, \partial''_k$. We claim that this η depends only on the choice of ζ. Indeed, due to exactness of rows in (2.12), a different choice of η_0 corresponds to a perturbation by an element in the image of $\mathscr{F}_{i+2}^{k-1}$ under Φ_{i+2}, which results in the new η_0 being in the same homology class. A different choice of ζ_0 corresponds to a perturbation by an element of $\partial_{k+1}(\mathscr{F}_i^{k+1})$ and an element of $\Phi_{i+1}(\mathscr{F}_{i+1}^k)$. The first of these is annihilated by the exactness of the bottom row of (2.12) and the way we defined η. Due to the commutativity of the diagram, the second perturbation corresponds to a perturbation of η_0 by an element of $\partial_k(\mathscr{F}_{i+1}^k)$, which yields the same homology class η.

Conversely, if one begins with an element $\eta \in \ker \partial''_{k-1}/\mathrm{im}\, \partial''_k$, for any representative $\eta_0 \in \mathscr{F}_{i+1}^{k-1}$, there is an element $\zeta_0 \in \mathscr{F}_i^k$ such that $\partial_k \zeta_0 = \Phi_{i+1}\eta_0$. This implies that ζ_0 corresponds to a homology class $\zeta \in \ker \partial'_k/\mathrm{im}\, \partial'_{k+1}$. We claim that this ζ depends only on η. Indeed, by the exactness of the bottom row, a different choice of ζ_0 is a perturbation by an element in $\partial_{k+1}(\mathscr{F}_i^{k+1})$, which corresponds to the same homology class ζ. A different choice of η_0 corresponds to a perturbation by an element in $\Phi_{i+2}(\mathscr{F}_{i+2}^{k-1})$ or in $\partial_k(\mathscr{F}_{i+1}^k)$. In the first case, the perturbation is annihilated by the time that we get to $\mathscr{F}_i^{k-1}$. In the second case, the commutativity of the diagram implies that the resulting perturbation of ζ_0 is an element in $\Phi_{i+1}(\mathscr{F}_{i+1}^k)$ and hence the homology class ζ is the same.

Clearly the above maps are inverses of each other, and hence we have the desired isomorphism. $\square$

Claim 2.13.

$$\frac{\ker \partial'_1}{\mathrm{im}\, \partial'_2} \cong \frac{\ker \Phi_{i+1}(0)}{\mathrm{im}\, \Phi_{i+2}(0)}.$$

Proof. As in Claim 2.12, we focus on a piece of (2.5):

$$
\begin{tikzcd}
\mathcal{F}^2_{i+2} \arrow[r, "\partial_2"] \arrow[d, "\Phi_{i+2}"'] & \mathcal{F}^1_{i+2} \arrow[r, "\partial_1"] \arrow[d, "\Phi_{i+2}"'] & \mathcal{F}^0_{i+2} \arrow[r, "\partial_{1/2}"] \arrow[d, "\Phi_{i+2}"'] & \mathcal{C}_{i+2} \arrow[r] \arrow[d, "\Phi_{i+2}(0)"'] & 0 \\
\mathcal{F}^2_{i+1} \arrow[r, "\partial_2"] \arrow[d, "\Phi_{i+1}"'] & \mathcal{F}^1_{i+1} \arrow[r, "\partial_1"] \arrow[d, "\Phi_{i+1}"'] & \mathcal{F}^0_{i+1} \arrow[r, "\partial_{1/2}"] \arrow[d, "\Phi_{i+1}"'] & \mathcal{C}_{i+1} \arrow[r] \arrow[d, "\Phi_{i+1}(0)"'] & 0 \\
\mathcal{F}^2_{i} \arrow[r, "\partial_2"] & \mathcal{F}^1_{i} \arrow[r, "\partial_1"] & \mathcal{F}^0_{i} \arrow[r, "\partial_{1/2}"] & \mathcal{C}_{i} \arrow[r] & 0.
\end{tikzcd}
\tag{2.13}
$$

Let ζ be an element in $\ker \partial_1'/\operatorname{im} \partial_2'$, let ζ_0 be a representative of ζ in $\mathscr{F}^1_i$. Then $\partial_1\zeta_0 \in \Phi_{i+1}(\mathscr{F}^0_{i+1})$ by assumption. Choose $\eta \in \mathscr{F}^0_{i+1}$ such that $\Phi_{i+1}\eta = \partial_1\zeta_0$, and then let $x_0 = \eta(0) \in \mathscr{C}_{i+1}$. By the way in which x_0 was defined and by the commutativity of (2.13), $x_0 \in \ker \Phi_{i+1}(0)$. Now let x be the homology class of x_0 in $\ker \Phi_{i+1}(0)/\operatorname{im} \Phi_{i+2}(0)$. We claim that this x depends only on ζ. First, by exactness, a different choice of η corresponds to a perturbation by an element in $\Phi_{i+2}(\mathscr{F}^0_{i+2})$. By the commutativity of (2.13), this corresponds to a perturbation by x_0 by an element in $\Phi_{i+2}(0)(\mathscr{C}_{i+2})$, which leaves x unaltered. By the exactness of rows, any perturbation of ζ_0 by an element of $\partial_2(\mathscr{F}^2_i)$ or $\Phi_{i+1}(\mathscr{F}^1_{i+1})$ is annihilated by the time one gets to $x_0 \in \mathscr{C}_{i+1}$.

Conversely, suppose that one starts with an element $x \in \ker \Phi_{i+1}(0)/\operatorname{im} \Phi_{i+2}(0)$. Choose a representative $x_0 \in \ker \Phi_{i+1}(0)$ for x. Let η_0 be a preimage of x_0 in $\mathscr{F}^0_{i+1}$. By the commutativity of (2.13), there exists $\zeta_0 \in \mathscr{F}^1_i$ such that $\partial_1\zeta_0 = \Phi_{i+1}\eta_0$. Hence, this ζ_0 is part of a homology class $\zeta \in \ker \partial_1'/\operatorname{im} \partial_2'$. We claim that ζ depends only on the choice of x. Indeed, by the exactness of the middle row in (2.13), a different choice of $\eta_0 \in \mathscr{F}^0_{i+1}$ corresponds to a perturbation by an element in $\partial_1(\mathscr{F}^1_{i+1})$. By the commutativity of (2.13), this corresponds to a perturbation of ζ_0 by an element of $\Phi_{i+1}(\mathscr{F}^1_{i+1})$, yielding the same homology class ζ. A different choice of ζ_0 so that $\partial_1\zeta_0 = \Phi_{i+1}\eta_0$ corresponds, by exactness of the bottom row of (2.13), to a perturbation by an element in $\partial_2(\mathscr{F}^2_i)$ which obviously yields the same homology class η. Finally, a different choice of x_0 corresponds to a perturbation by an element in $\Phi_{i+2}(0)(\mathscr{C}_{i+2})$. By the exactness of the top row and second to last column of (2.13), one sees that this perturbation is annihilated by the time one arrives at $\mathscr{F}^0_i$.

Clearly the above two maps are inverses of each other. Hence the desired isomorphism is established. $\square$

We can now establish the statement of the theorem. Labeling more explicitly now, we let

$$\cdots \overset{\partial_2^i}{\to} \mathscr{H}^1_i \overset{\partial_1^i}{\to} \mathscr{H}^0_i \to 0$$

be the Koszul complex of $\mathscr{H}_i$. For $k \geqslant 2$, Claim 2.12 yields the following sequence of isomorphisms:

$$\frac{\ker \partial_k^0}{\operatorname{im} \partial_{k+1}^0} \cong \frac{\ker \partial_{k-1}^1}{\operatorname{im} \partial_k^1} \cong \cdots \frac{\ker \partial_1^{k-1}}{\operatorname{im} \partial_2^{k-1}}. \tag{2.14}$$

We then use Claim 2.13 to finish off the sequence:

$$\frac{\ker \partial_1^{k-1}}{\operatorname{im} \partial_2^{k-1}} \cong \frac{\ker \Phi_k(0)}{\operatorname{im} \Phi_{k+1}(0)}. \tag{2.15}$$

Together, (2.14) and (2.15) establish the statement of the theorem. □

Our goal is to describe the homology of the localized complex

$$\cdots \xrightarrow{\Phi_3(\lambda)} \mathscr{C}_2 \xrightarrow{\Phi_2(\lambda)} \mathscr{C}_1 \xrightarrow{\Phi_1(\lambda)} \mathscr{C}_0.$$

for an arbitrary $\lambda \in B_d$. We will attain this goal in Section 4, but we need some machinery first. This is the subject of the next section.

3. The Möbius transform

In this section, we define the notion of a Möbius transform of a contractive $\mathscr{A}_d$-module. The use of Möbius transforms in multivariable operator theory is by no means new. See [6, Section 4], for instance.

We begin by summarizing the main properties of Möbius transforms on the unit ball in $\mathbb{C}^d$. For a more detailed exposition, we refer the reader to [9]. Recall that a Möbius transform on the unit ball in $\mathbb{C}^d$ is a continuous bijection $\varphi : \bar{B}_d \to \bar{B}_d$ which satisfies the following (somewhat redundant) properties:

(a) φ is holomorphic in B_d;
(b) $\varphi(B_d) = B_d$;
(c) $\varphi(\partial B_d) = \partial B_d$.

Let $\lambda \in B_d$, and define φ_λ as follows:

$$\varphi_\lambda(z_1, z_2, \ldots, z_d) = \left(\frac{(1 - |\lambda|^2) z_1}{1 - \sum_{k=1}^d \overline{\lambda_k} z_k} - \lambda_1, \ldots, \frac{(1 - |\lambda|^2) z_d}{1 - \sum_{k=1}^d \overline{\lambda_k} z_k} - \lambda_d \right) \tag{3.1}$$

or in vector notation,

$$\varphi_\lambda(z) = \frac{P_{\mathbb{C}\lambda}(z - \lambda) + (1 - |\lambda|^2) P_{\mathbb{C}\lambda}^{\perp} z}{1 - \langle z, \lambda \rangle}, \quad z \in B_d,$$

where $P_{\mathbb{C}\lambda}$ is the orthogonal projection onto the one-dimensional subspace $\mathbb{C}\lambda \subseteq \mathbb{C}^d$. Some calculations reveal that φ_λ is a Möbius transform and that $\varphi_\lambda(\lambda) = 0$. It is a striking fact that *any* Möbius transform φ can be written in the form $u \circ \varphi_\lambda$, where u is a unitary operator on $\mathbb{C}^d$ and $\lambda = \varphi^{-1}(0)$. This fact allows us to find a useful formula for the expression $\langle \varphi(w), \varphi(z) \rangle$, where $z, w \in B_d$. Calculating this first for the case where $\varphi = \varphi_\lambda$, we obtain the following identity:

$$\langle \varphi(z), \varphi(w) \rangle = 1 - \frac{(1-|\lambda|^2)(1-\langle z, w \rangle)}{(1-\langle \lambda, w \rangle)(1-\langle z, \lambda \rangle)}, \quad z, w \in B_d. \tag{3.2}$$

Consequently, for any Möbius transform φ with $\lambda = \varphi^{-1}(0)$, $\langle \varphi(w), \varphi(z) \rangle$ is also given by (3.2).

The following theorem unveils the role that is played by Möbius transforms in the theory of contractive $\mathscr{A}_d$-modules.

Theorem 3.1. *Let $\varphi^1, \varphi^2, \ldots, \varphi^d$ be the coordinates of the Möbius transform φ. Define the map $\Phi : \overbrace{H_d^2 \oplus \cdots H_d^2}^{d \text{ times}} \to H_d^2$ to be left multiplication by the row vector $(\varphi^1 \cdots \varphi^d)$, i.e.*

$$\Phi \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_d \end{pmatrix} = \sum_{k=1}^{d} \varphi^k \xi_k, \quad \xi_1, \xi_2, \ldots, \xi_d \in H_d^2.$$

Then Φ is a partially isometric module homomorphism with range $\{\xi \in H_d^2 : \xi(\lambda) = 0\} = \{k_\lambda\}^\perp$.

Proof. It is obvious that Φ is a module homomorphism. To show that it is partially isometric with the stated range, it suffices to show that

$$\langle (1 - \Phi\Phi^*) k_w, k_z \rangle = \langle P_\lambda k_w, k_z \rangle \tag{3.3}$$

for any $w, z \in B_d$, where P_λ is the orthogonal projection onto the space $\mathbb{C}k_\lambda$. The sufficiency of this condition follows from the fact that the set of all k_z's forms a spanning set of H_d^2.

We compute the left-hand side of (3.3). Using (1.1), (1.2), and formula (3.2), we have

$$\langle (1 - \Phi\Phi^*) k_w, k_z \rangle = \frac{1}{1 - \langle z, w \rangle} - \frac{\langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle}$$

$$= \frac{1}{1 - \langle z, w \rangle} - \frac{1}{1 - \langle z, w \rangle} + \frac{1 - |\lambda|^2}{(1 - \langle \lambda, w \rangle)(1 - \langle z, \lambda \rangle)}$$
$$= \frac{1 - |\lambda|^2}{(1 - \langle \lambda, w \rangle)(1 - \langle z, \lambda \rangle)}. \tag{3.4}$$

We compute the right-hand side of (3.3) as follows:

$$\langle P_\lambda k_w, k_z \rangle = \frac{\langle k_w, k_\lambda \rangle}{||k_\lambda||^2} \langle k_\lambda, k_z \rangle = \frac{1 - |\lambda|^2}{(1 - \langle \lambda, w \rangle)(1 - \langle z, \lambda \rangle)}, \tag{3.5}$$

which is identical to (3.4), hence we have established (3.3). □

We are now in a position to define a Möbius transform of H_d^2.

Definition 3.2. Let φ be a Möbius transform. We define $(H_d^2)_\varphi$ to be the $\mathscr{A}_d$-module whose underlying Hilbert space is H_d^2 and whose $\mathscr{A}_d$-module structure is given by $z_i \cdot \xi = \varphi^i \xi$ for any $\xi \in H_d^2$ and $i = 1, 2, \ldots, d$, where φ^i is the ith coordinate function of φ.

Theorem 3.3. *Let φ be a Möbius transform. Then $(H_d^2)_\varphi$ is a contractive $\mathscr{A}_d$-module. Furthermore, if we set $\lambda = \varphi^{-1}(0)$, then the map $U_\varphi : H_d^2 \to (H_d^2)_\varphi$ defined by $U_\varphi \xi = (\xi \circ \varphi) \frac{k_\lambda}{||k_\lambda||}$ is a unitary module isomorphism.*

Proof. Theorem 3.1 implies that $(H_d^2)_\varphi$ is contractive as an $\mathscr{A}_d$-module. Hence by Theorem 1.3 there exists a minimal dilation $U : (H_d^2 \otimes \mathscr{D}) \otimes \mathscr{S} \to (H_d^2)_\varphi$. We claim that $\mathscr{S}$ is trivial. This follows from a result of Arveson [2] which states that the minimal dilation of a contractive $\mathscr{A}_d$-module has trivial spherical part iff the following condition is satisfied for the associated d-tuple $(\Phi_1, \Phi_2, \ldots, \Phi_d)$:

$$\mathrm{WOT} - \lim_{n \to \infty} \sum_{i_1, i_2, \ldots, i_n}^{d} \Phi_{i_1} \cdots \Phi_{i_n} \Phi_{i_n}^* \cdots \Phi_{i_1}^* = 0.$$

Since $(\Phi_1 \ \Phi_2 \ \cdots \ \Phi_d)$ is a contraction, it follows that the sequence $\sum_{i_1, i_2, \ldots, i_n}^{d} \Phi_{i_1} \Phi_{i_2} \cdots \Phi_{i_n} \Phi_{i_n}^* \cdots \Phi_{i_1}^*$ is uniformly bounded. Since the set $\{k_\lambda : \lambda \in B_d\}$ generates H_d^2, it suffices to show that

$$\mathrm{WOT} - \lim_{n \to \infty} \sum_{i_1, i_2, \ldots, i_n}^{d} \langle \Phi_{i_1} \cdots \ \Phi_{i_n} \Phi_{i_n}^* \cdots \Phi_{i_1}^* k_\lambda, k_z \rangle = 0$$

for any $\lambda, z \in B_d$. By (1.2), we have

$$\sum_{i_1,\dots,i_n}^{d} \langle \Phi_{i_1} \cdots \Phi_{i_n} \Phi_{i_n}^* \cdots \Phi_{i_1}^* k_\lambda, k_z \rangle = \langle \Phi_{i_n}^* \cdots \Phi_{i_1}^* k_\lambda, \Phi_{i_n}^* \cdots \Phi_{i_1}^* k_z \rangle$$
$$= \overline{\varphi^{i_1}(\lambda)} \cdots \overline{\varphi^{i_n}(\lambda)} \varphi^{i_n}(z) \cdots \varphi^{i_1}(z) \langle k_\lambda, k_z \rangle = \langle \varphi(z), \varphi(\lambda) \rangle_{\mathbb{C}^d}^n. \tag{3.6}$$

Since $|\langle \varphi(z)\varphi(\lambda)\rangle_{\mathbb{C}^d}| \leqslant ||\varphi(\lambda)||_{\mathbb{C}^d}^2 ||\varphi(z)||_{\mathbb{C}^d}^2 < 1$, it follows that (3.6) tends to 0 as n tends to infinity.

We now show that the Hilbert space $\mathscr{C}$ in the minimal dilation $U : H_d^2 \otimes \mathscr{C} \to (H_d^2)_\varphi$ can be taken to be $\mathbb{C}$ and $U(\xi \otimes 1) = (\xi \circ \varphi)\frac{k_\lambda}{||k_\lambda||}$. Indeed, if we let $(S_1, S_2, \dots, S_d)$ be the d-tuples associated with H_d^2, then by Theorem 3.1, $P_0 = 1 - \sum_{k=1}^d S_k S_k^*$ and $P_\lambda = 1 - \sum_{k=1}^d \Phi_k \Phi_k^*$. Since U is a coisometric module homomorphism, it follows that

$$U(P_0^\perp \otimes 1_{\mathscr{C}})U^* = U\left(1 - \sum_{k=1}^d (S_k \otimes 1_{\mathscr{C}})(S_k \otimes 1_{\mathscr{C}})^*\right)U^*$$
$$= 1 - \sum_{k=1}^d \Phi_k \Phi_k^* = P_\lambda^\perp. \tag{3.7}$$

Hence $(P_0^\perp \otimes 1_{\mathscr{C}})U^*$ is a rank 1 operator. Let $\eta \in \mathscr{C}$ be such that $k_0 \otimes \eta$ is orthogonal to $U^*(H_d^2)_\varphi$. Hence for all elements W in the nonunital algebra generated by the $S_k \otimes 1_{\mathscr{C}}$'s and for every $\xi \in (H_d^2)_\varphi$, $0 = \langle W^*(k_0 \otimes \eta), U^*\xi \rangle = \langle k_0 \otimes \eta, WU^*\xi \rangle$. But by minimality $H_d^2 \otimes \mathscr{C}$ is generated by elements of the form $WU^*\xi$. Hence $\eta = 0$. It follows that $\mathscr{C}$ is one dimensional, hence we may take it to be $\mathbb{C}$. Identifying $H_d^2 \otimes \mathbb{C}$ with H_d^2, and using (3.7), we may assume that $Uk_0 = ck_\lambda$ where c is a positive number. The module property implies that for any $\xi \in H_d^2$, $U\xi = (\xi \circ \varphi)ck_\lambda$. We will complete the proof by showing that U is a unitary operator, and hence $c = 1$. Since U is already coisometric, it suffices to show that $\ker U = 0$. To this end, suppose $\xi \in H_d^2$ and $(\xi \circ \varphi)ck_\lambda = U\xi = 0$. Since k_λ is non-zero, the function $\xi \circ \varphi$ must then be zero. But φ is a bijection on B_d, hence $\xi = 0$. Thus $\ker U$ is trivial, and U is a unitary operator. $\square$

The following corollary shows how a Möbius transform U_φ provides a means of changing the base point from 0 to $\lambda \in B_d$ when considering the module H_d^2.

Corollary 3.4. *Let φ be a Möbius transform and let $\lambda = \varphi^{-1}(0)$. Then $U_\varphi\{k_0\}^\perp = \{k_\lambda\}^\perp$.*

Proof. This is obvious from the definition of U_φ.

To conclude this section, we demonstrate an ergodicity property of the set of Möbius transforms.

Theorem 3.5. *Let $\mathscr{M}$ be a proper nontrivial closed submodule of H^2_d. Then there exists $\xi \in \mathscr{M}$ and a Möbius transform φ such that $U_\varphi \xi \notin \mathscr{M}$.*

Proof. Since $\mathscr{M}$ is a proper closed submodule, it cannot contain k_0. Hence

$$M = \sup\{|\langle \xi, k_0 \rangle| : ||\xi|| = 1, \xi \in \mathscr{M}\} < 1.$$

By Theorem 3.2 in [8], there exists $\lambda \in B_d$ such that

$$\frac{||P_{\mathscr{M}} k_\lambda||}{||k_\lambda||} > M. \tag{3.8}$$

An explicit calculation involving (3.1) shows that

$$U_{\varphi_{-\lambda}} U_{\varphi_\lambda} \xi = \xi \circ u,$$

where u is a unitary operator on $\mathbb{C}^d$. Hence by the definition of U_u, $U_{\varphi_{-\lambda}} U_{\varphi_\lambda} = U_u$. Hence $U^*_{\varphi_\lambda} = U_{u^*} U_{\varphi_{-\lambda}}$. Therefore by our assumption that $\mathscr{M}$ is invariant under Möbius transforms, it follows that

$$\sup\{|\langle U^*_{\varphi_\lambda} \xi, k_0 \rangle| : ||\xi|| = 1, \xi \in \mathscr{M}\} \leqslant M. \tag{3.9}$$

By definition, $U_{\varphi_\lambda} k_0 = \frac{k_\lambda}{||k_\lambda||}$. Hence by (3.9), $|\langle \xi, \frac{k_\lambda}{||k_\lambda||} \rangle| \leqslant M$ for all $\xi \in \mathscr{M}$ such that $||\xi|| = 1$. But this implies that

$$\frac{||P_{\mathscr{M}} k_\lambda||}{||k_\lambda||} \leqslant M,$$

which contradicts (3.8). $\square$

4. The homology of localized free resolutions

In this section, we use the machinery developed in Section 3 to extend Theorem 2.11. We first generalize Definition 3.2.

Definition 4.1. Let $\mathscr{H}$ be a pure contractive $\mathscr{A}_d$-module, and let φ be a Möbius transform. We define the Möbius transform of $\mathscr{H}$ by φ to be the module $(\mathscr{H})_\varphi$ with underlying Hilbert space $\mathscr{H}$ and $\mathscr{A}_d$ action defined by $z_i \cdot \xi = \varphi^i(T_1, T_2, \ldots, T_d)\xi$ for all $\xi \in H^2_d$ and $i = 1, 2, \ldots, d$.

Theorem 4.2. *The Möbius transform of a spherical module $\mathscr{S}$ is itself spherical.*

Proof. Let $(Z_1, Z_2, \ldots, Z_d)$ be the d-tuple associated with $\mathscr{S}$. Since $\mathscr{S}$ is spherical, there exists a C^*-algebra homomorphism $\pi : C^*(\partial B_d) \to C^*(Z_1, \ldots, Z_d)$ such that $\pi(z_i) = Z_i$ for each i. Hence $\pi(\varphi^i) = \varphi^i(Z_1, \ldots, Z_d)$ for each i. But $\sum_{i=1}^d \varphi^i \overline{\varphi^i} = 1$, thus $\sum_{i=1}^d \varphi^i(Z_1, \ldots, Z_d)\ \varphi^i(Z_1, \ldots, Z_d)^* = \pi(\sum_{i=1}^d \varphi^i \overline{\varphi^i}) = 1$. Clearly, the C^*-algebra generated by the φ^i's is abelian, thus $(\mathscr{S})_\varphi$ is spherical. $\square$

The main result of this section is the following:

Theorem 4.3. *Let $\mathscr{H}$ be a contractive $\mathscr{A}_d$-module, and let φ be a Möbius transform with $\lambda = \varphi^{-1}(0)$. Let*

$$\cdots \overset{\partial_3'}{\to} E_2 \overset{\partial_2'}{\to} E_1 \overset{\partial_1'}{\to} E_0 \to 0$$

be the Koszul complex of $(\mathscr{H})\varphi$, and let

$$\cdots \overset{\Phi_3(\lambda)}{\to} \mathscr{C}_2 \overset{\Phi_2(\lambda)}{\to} \mathscr{C}_1 \overset{\Phi_1(\lambda)}{\to} \mathscr{C}_0 \tag{4.1}$$

be the localization at λ of a free resolution of $\mathscr{H}$. Then for $k \geqslant 1$,

$$\frac{\ker \partial_k'}{\operatorname{im} \partial_{k+1}'} \cong \frac{\ker \Phi_k(\lambda)}{\operatorname{im} \Phi_{k+1}(\lambda)}.$$

Proof. Let

$$\cdots \overset{\Phi_2}{\to} (H_d^2 \otimes \mathscr{C}_1) \oplus \mathscr{S}_1 \overset{\Phi_1}{\to} (H_d^2 \otimes \mathscr{C}_0) \oplus \mathscr{S}_0 \overset{\Phi_0}{\to} \mathscr{H} \to 0 \tag{4.2}$$

be the free resolution of $\mathscr{H}$ whose localization to λ is (4.1). Since Φ_i is a module homomorphism it follows that for $i \geqslant 0$, $\varphi^j \Phi_i = \Phi_i \varphi^j$ for $j = 1, 2, \ldots, d$. Clearly $((H_d^2 \otimes \mathscr{C}_i) \oplus \mathscr{S}_i)_\varphi = ((H_d^2)_\varphi \otimes \mathscr{C}_i) \oplus (\mathscr{S}_i)_\varphi)$, Hence have the following reexpression of (4.2):

$$\cdots \overset{\Phi_1}{\to} ((H_d^2)_\varphi \otimes \mathscr{C}_0) \oplus (\mathscr{S}_0)_\varphi \overset{\Phi_0}{\to} (\mathscr{H})_\varphi \to 0.$$

For each $i \geqslant 1$ let $W_\varphi = (U_\varphi \otimes 1_{\mathscr{C}_i}) \oplus 1_{\mathscr{S}_i}$ and let $\Phi_i' = W_\varphi^* \Phi_i W_\varphi$. Then we have the following partial isomorphism of complexes:

$$\begin{tikzcd}
\cdots \arrow[r, "\Phi_1"] & ((H_d^2)_\varphi \otimes \mathcal{C}_0) \oplus (\mathcal{S}_0)_\varphi \arrow[r, "\Phi_0"] \arrow[d, "W_\varphi^*"'] & (\mathcal{H})_\varphi \arrow[r] & 0 \\
\cdots \arrow[r, "\Phi_1'"] & (H_d^2 \otimes \mathcal{C}_0) \oplus (\mathcal{S}_0)_\varphi. & &
\end{tikzcd}$$

Since two isomorphic complexes have equivalent homologies, there is a coisometric homomorphism $\Phi'_0 : (H^2_d \otimes \mathscr{C}_0) \oplus (\mathscr{S}_0)_\varphi \to (\mathscr{H})_\varphi$ which makes the following sequence exact:

$$\cdots \xrightarrow{\Phi'_1} (H^2_d \otimes \mathscr{C}_0) \oplus (\mathscr{S}_0)_\varphi \xrightarrow{\Phi'_0} (\mathscr{H})_\varphi \to 0.$$

By Theorem 4.2, this is a free resolution for $(\mathscr{H})_\varphi$, hence Theorem 2.11 implies that

$$\frac{\ker \partial'_k}{\operatorname{im} \partial'_{k+1}} \cong \frac{\ker \Phi'_k(0)}{\operatorname{im} \Phi'_{k+1}(0)}$$

for $k \geqslant 1$. We now compute $\Phi'_i(0)$. Recall that for $\lambda \in B_d$, $\Phi(\lambda)$ (resp. $\Phi'(\lambda)$) is the localization of $P_{H^2_d \otimes \mathscr{C}_{i-1}} \phi_i \restriction_{H^2_d \otimes \mathscr{C}_i}$ (resp. $P_{H^2_d \otimes \mathscr{C}_{i-1}} \Phi'_i \restriction_{H^2_d \otimes \mathscr{C}_i}$). Let $\eta \in \mathscr{C}_i$ and $\eta' \in \mathscr{C}_{i+1}$. Then

$$\langle \Phi'_i(0)\eta, \eta' \rangle = \langle \Phi'_i(k_0 \otimes \eta), k_0 \otimes \eta' \rangle = \langle W^*_\varphi \Phi_i W_\varphi (k_0 \otimes \eta), k_0 \otimes \eta' \rangle$$

$$= \left\langle \Phi_i\left(\frac{k_\lambda}{||k_\lambda||} \otimes \eta\right), \frac{k_\lambda}{||k_\lambda||} \otimes \eta' \right\rangle = \left\langle \frac{k_\lambda}{||k_\lambda||} \otimes \eta, \Phi^*_i\left(\frac{k_\lambda}{||k_\lambda||} \otimes \eta'\right)\right\rangle$$

$$= \left\langle \frac{k_\lambda}{||k_\lambda||} \otimes \eta, \frac{k_\lambda}{||k_\lambda||} \otimes \Phi_i(\lambda)^* \eta' \right\rangle = \frac{||k_\lambda||^2 \langle \eta, \Phi_i(\lambda)^* \eta' \rangle}{||k_\lambda||^2} = \langle \Phi_i(\lambda)\eta, \eta' \rangle.$$

Since η and η' were arbitrary, $\Phi'_i(0) = \Phi_i(\lambda)$ for each $i \geqslant 1$. The conclusion of the theorem now follows. $\square$

Corollary 4.4. *Let $\mathscr{H}$ be a contractive $\mathscr{A}_d$-module, and let*

$$\cdots \xrightarrow{\Phi_2} \mathscr{F}_1 \xrightarrow{\Phi_1} \mathscr{F}_0 \xrightarrow{\Phi_0} \mathscr{H} \to 0 \tag{4.3}$$

be a free resolution of $\mathscr{H}$. Let $\lambda \in B_d$, and let

$$\cdots \xrightarrow{\Phi_3(\lambda)} \mathscr{C}_2 \xrightarrow{\Phi_2(\lambda)} \mathscr{C}_1 \xrightarrow{\Phi_1(\lambda)} \to \mathscr{C}_0$$

be the localized complex of (4.3) *at λ. Then for $k \geqslant d+1$,* $\ker \Phi_k(\lambda) = \operatorname{im} \Phi_{k+1}(\lambda)$.

References

[1] W. Arveson, The Dirac operator of a commuting d-tuple, pre-print.

[2] W. Arveson, Subalgebras of C^*-algebras III. Multivariable operator theory, Acta Math. 181 (2) (1998) 159–228.

[3] W. Arveson, The curvature invariant of a Hilbert module over $\mathbf{c}[z_1, \ldots, z_d]$, J. Reine Angew. Math. 522 (2000) 173–236.

[4] J. Conway, A course in functional analysis, 1990.

[5] R.E. Curto, Fredholm and invertible n-tuples of operators. The deformation problem, Trans. Amer. Math. Soc. 266 (1) (1981) 129–159.
[6] K.R. Davidson, D.R. Pitts, The algebraic structure of non-commutative analytic Toeplitz algebras, Math. Ann. 311 (2) (1998) 275–303.
[7] D. Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry, Springer, New York, 1995.
[8] D. Greene, S. Richter, C. Sundberg, The structure of inner multipliers on spaces with complete Nevanlinna Pick kernels, J. Funct. Anal., to appear.
[9] W. Rudin, Function theory in the unit ball of $\mathbb{C}^n$, Springer, New York, Berlin, 1980.
[10] J.L. Taylor, A joint spectrum for several commuting operators, J. Funct. Anal. 6 (1970) 172–191.
[11] F.-H. Vasilescu, A characterization of the joint spectrum in Hilbert spaces, Rev. Roumaine Math. Pures Appl. 22 (7) (1977) 1003–1009.

Further reading

W. Arveson, The curvature of a Hilbert module over $\mathbb{C}[z_1, z_2, \ldots, z_d]$, Proc. Natl. Acad. Sci. USA 96 (20) (1999) 11096–11099.
R. Hartshorne, Algebraic Geometry, in: Graduate Texts in Mathematics, Vol. 52, Springer, New York, 1977.
S. Lang, Algebra, 2nd Edition, Addison–Wesley Publishing Company Advanced Book Program, Reading, MA, 1984.
S. Parrott, The curvature of a single contraction operator on a Hilbert space, pre-print.
J.J. Rotman, An Introduction to Homological Algebra, Academic Press Inc., Harcourt Brace Jovanovich Publishers, New York, 1979.
J.L. Taylor, The analytic-functional calculus for several commuting operators, Acta Math. 125 (1970) 1–38.